Topologist's Sine Curve

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Let $\Gamma = \{(x, y) : 0 < x \leq 1,\ y = \sin(\frac{1}{x})\} \cup \{(0, y) : |y| \leq 1\}$

Theorem 1. Γ *is not path connected.*

Proof. Suppose $f(t) = (a(t), b(t))$ is a continuous curve defined on $[0, 1]$ with $f(t) \in \Gamma$ for all t and $f(0) = (0, 0), f(1) = (\frac{1}{\pi}, 0)$. Then by the intermediate value theorem there is a $0 < t_1 < 1$ so that $a(t_1) = \frac{2}{3\pi}$. Then there is $0 < t_2 < t_1$ so that $a(t_2) = \frac{2}{5\pi}$. Continuing, we get a decreasing sequence t_n so that $a(t_n) = \frac{2}{(2n+1)\pi}$. It follows that $b(t_n) = (-1)^n$. Now since t_n is a decreasing sequence bounded from below it tends to limit $t_n \to c$. Since f is continuous $\lim f(t_n)$ must exist. But $\lim_{n\to\infty} b(t_n)$ does not exist. $\square$